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SINGLE-FREQUENCY OSCILLATIONS OF NON-LINEAR SYSTEMS WITH DISTRIBUTED PARAMETERS†

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A weakly non-linear oscillatory system with distributed parameters is investigated. An asymptotic method of constructing a solution, which describes the oscillatory motions of the single-mode (single-frequency) approximation, which is usually implemented in practical problems, is described and justified. Constructive sufficient conditions are formulated and the closeness of the approximate single-frequency solution to the exact solution in an asymptotically long time interval is proved. Possible extensions of the structure of the perturbing functions are considered and the case of the finite-mode approximation is investigated. Solutions of specific problems, which are of practical interest, are constructed to illustrate the effectiveness of the single-frequency approximation method. © 1999 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a weakly non-linear oscillatory system, described by a hyperbolic-type perturbed equation with homogeneous boundary conditions of the third kind

$$\ddot{u} = u'' + \varepsilon f(x, u, u', \dot{u}), \quad u = u(x, t), \quad 0 < x < 1 \tag{1.1}$$

$$\alpha_0 u'(0,t) - \beta_0 u(0,t) = 0, \quad \alpha_1 u'(1,t) + \beta_1 u(1,t) = 0 \tag{1.2}$$

$$\alpha_{0,1} \geqslant 0, \quad \beta_{0,1} \geqslant 0, \quad \alpha_{0,1} + \beta_{0,1} = 1, \quad t \geqslant 0$$

The dot denotes a derivative with respect to dimensionless time t, and the prime denotes the normalized space coordinate x. The parameter ε, $0 \leqslant \varepsilon \leqslant \varepsilon_0$ ($\varepsilon_0 \ll 1$) represents the effect of non-linear perturbing factors, described by the function f of fairly general form. The boundary conditions of the third kind (1.2) with normalized coefficients $\alpha_{0,1}$, $\beta_{0,1}$ take into account the flexibility with which the elastic system is attached at its ends (when $\alpha_{0,1} > 0$). In the limiting cases $\beta_{0,1} = 1$ ($\alpha_{0,1} = 1$) or $\beta_{0,1} = 0$ ($\beta_{0,1} = 0$) ($\alpha_{0,1} = 1$) we have boundary conditions of the first or second kinds at one or both ends.

For the oscillatory system described by boundary-value problem (1.1), (1.2), we have the following Cauchy problem with respect to the time t

$$u(x,0) = h(x), \quad \dot{u}(x,0) = g(x), \quad 0 < x < 1 \tag{1.3}$$

where the functions $h(x)$ and $g(x)$ define the distributions of the displacements u and the velocities $\dot{u}$ at the initial instant of time $t = 0$. Note that system (1.1), (1.2) is autonomous, i.e. it does not contain the time explicitly. In what follows, the functions f, h and g are assumed to be sufficiently smooth and such that a strong (physical) solution $u(x, t, \varepsilon)$ of problem (1.1)–(1.3) exists (the functions u, u' and $\dot{u}$ are square integrable with respect to x) in an asymptotically long time interval $t \sim \varepsilon^{-1}$.

When there are no perturbations ($\varepsilon = 0$) we have a classical initial-boundary-value problem with boundary conditions of the third kind. This problem has been investigated fairly fully in courses on mathematical physics [1–3]. In the general case, its solution $u_0(x, t)$ is an almost periodic function of time with a denumerable basis $\{\nu_n\}$ of incommensurable frequencies ν_n. This solution can be represented in the form of a Fourier series in an orthonormal system of functions (basis) $\{X_n(x)\}$

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$$u_0(x,t)=\sum_{n=1}^{\infty} X_n(x)\theta_n(t)\equiv(X(x),\theta(t)),\quad X_n(x)=\frac{\chi_n(x)}{\|\chi_n\|}$$

$$\chi_n(x)=\beta_0\sin\nu_n x+\alpha_0\nu_n\cos\nu_n x,\quad \|\chi_n\|^2=\int_0^1\chi_n^2(x)dx,\quad n=1,2,\dots$$

$$\theta_n(t)=h_n\cos\nu_n t+(g_n/\nu_n)\sin\nu_n t \tag{1.4}$$

$$\nu_n=\operatorname*{Arg}_{\nu}[(\beta_0\beta_1-\nu^2\alpha_0\alpha_1)\sin\nu+(\alpha_0\beta_1+\beta_0\alpha_1)\nu\cos\nu]>0$$

$$h_n=\langle h,X_n\rangle\equiv\int_0^1 h(x)X_n(x)dx,\quad g_n=\langle g,X_n\rangle\equiv\int_0^1 g(x)X_n(x)dx$$

The properties of the convergence of the series for u_0(1.4) and the derivatives are determined by the rate at which the coefficients h_n and g_n decrease as $n\to\infty$.

It follows from Eq. (1.4) for the eigenfrequencies ν_n that $\nu_n\to\pi n$ when $\alpha_0\alpha_1>0$ and $\nu_n\to\pi(n+1/2)$ when $\alpha_0\alpha_1=0$, $\alpha_0\beta_1+\beta_0\alpha_1>0$. This property of the spectrum leads to extremely complex behaviour of the function f with time, after substituting the unperturbed solution $u_0(x,t)$ and its derivatives into it. The problem of the existence of a uniform mean with respect to t and the absence of "internal" resonance and "small denominators" is the main difficulty when using perturbation theory. The use of the standard formal approach to constructing an approximate solution (see [4] and the bibliography) involves, in general, satisfying a number of limiting non-constructive conditions which, obviously, can only be done for linear functions f. The problem of the convergence of the Fourier series, representing the approximate formal solution and its derivatives, also arises here. The lack of a due basis gives rise to certain difficulties for the reliable use of the proposed solution algorithms as well as doubts in estimating the reliability of the results obtained when investigating specific oscillatory systems.

A more promising theoretical approach to investigating complex multifrequency systems is the asymptotic single-frequency approximation method [4–7] and the related methods of averaging (separation of motions) [4–8] and local integral manifolds [4–6, 9]. In applied investigations it has been established that, in linear systems with distributed parameters, single-frequency oscillations of the fundamental (lowest) mode occur [10]. Higher-mode oscillations are not excited in practice and, moreover, they decay very rapidly. These observations require a theoretical basis using constructive sufficient conditions.

2. INVESTIGATION OF AN AUXILIARY DENUMERABLE SYSTEM

Our calculations so far have mainly been of an auxiliary formal nature. They hold when the requirements imposed on the structure and smoothness of the functions f, h and g are extremely simple.

We will seek a solution of the perturbed problem (1.1)–(1.3) in the form of a Fourier series in an orthonormal complete system of functions $\{X_n(x)\}$

$$u(x,t,\varepsilon)=\sum_{n=1}^{\infty}X_n(x)\theta_n(t,\varepsilon)\equiv(X,\theta) \tag{2.1}$$

in which the functions $X_n(x)$ are defined by (1.4) while θ_n are unknown generalized coordinates of the system. After substituting (2.1) into Eq. (1.1), using the standard procedure for determining the required variables θ_n we obtain the denumerable Cauchy problem [4]

$$\ddot\theta_n+\nu_n^2\theta_n=\varepsilon f_n(\theta,\dot\theta),\quad \theta_n(0,\varepsilon)=h_n,\quad \dot\theta_n(0,\varepsilon)=g_n$$
$$\theta=(\theta_1,\theta_2,\dots,\theta_n,\dots),\quad \dot\theta=(\dot\theta_1,\dot\theta_2,\dots,\dot\theta_n,\dots) \tag{2.2}$$
$$f_n(\theta,\dot\theta)=\langle f(x,(X(x),\theta),\ (X'(x),\theta),\ (X(x),\dot\theta)),\ X_n(x)\rangle$$

Instead of θ_n, $\dot\theta_n$ we will introduce slow (osculating) variables a_n, b_n which are similar to the Van der Pol variables [6]. We obtain a standard denumerable Cauchy problem (in the Krylov–Bogolyubov sense) of the form

$$\dot{a}_n = \frac{\varepsilon}{\nu_n} F_n(a,b,\varphi)\cos\varphi_n, \quad \dot{b}_n = -\frac{\varepsilon}{\nu_n} F_n(a,b,\varphi)\sin\varphi_n, \quad n = 1,2,\ldots$$

$$F_n(a,b,\varphi) \equiv f_n(\theta,\dot{\theta}), \quad \varphi = (\varphi_1,\varphi_2,\ldots,\varphi_n,\ldots) \tag{2.3}$$

$$\theta_n = a_n \sin\varphi_n + b_n\cos\varphi_n, \quad \dot{\theta}_n = \nu_n(a_n\cos\varphi_n - b_n\sin\varphi_n)$$

$$\varphi_n = \nu_n t, \quad a_n(0,\varepsilon) = a_n^0 = g_n/\nu_n, \quad b_n(0,\varepsilon) = b_n^0 = h_n$$

Note that the functions F_n contain as the arguments the expressions $\nu_m a_m$ and $\nu_m b_m$ and, generally speaking, do not satisfy the Lipschitz condition with respect to a and b, since $\nu_m \sim m \to \infty$. The right-hand sides of Eqs (2.3) will be extremely complex almost periodic functions of t with a denumerable basis of frequencies $\{\nu_n\}$ (1.4). This behaviour of the frequencies of the partial oscillations ν_n as $n \to \infty$ leads to the well-known difficulties mentioned in Section 1, related to the problem of "small denominators", which are increased by the unlimited dimension of system (2.3). The existence of a uniform mean with respect to t of these functions and the properties of smoothness of the means of both the functions a and b are the main difficulties in analysing the continuous non-linear function f of u, u', $\dot{u}$ in the general case. Moreover, the equations for all the components a_n and b_n will be coupled, which makes their analysis quite impossible. These complications make it difficult to use the formal scheme of the method of averaging, the method of "truncation" of the denumerable system and a number of other results [4].

3. CONSTRUCTION OF THE SINGLE-FREQUENCY APPROXIMATION

Further, as in the single-frequency (single-mode) Krylov–Bogolyubov–Mitropol'skii approximation method, we will assume that the following constructively verifiable non-formal conditions are satisfied [4–7, 10].

1. The initial distributions of the displacements $h(x)$ and the velocities $g(x)$ (1.3) satisfy the necessary condition for single-frequency oscillations

$$h_1^2 + g_1^2 > 0, \quad h_n^2 + g_n^2 = 0, \quad n = 2,3,\ldots \tag{3.1}$$

This means that $a_n^0 = b_n^0 = 0$, $n \geq 2$, i.e. the initial distribution is proportional to the first mode of the oscillations of the unperturbed system (1.1). In a relatively short time interval $t \sim 1$, the motion of system (1.1)–(1.3), (3.1) will be close to $X_1(x)\theta_1(t)$, since $a_1 = a_1^0 + O(\varepsilon)$, $b_1 = b_1^0 + O(\varepsilon)$, $a_n, b_n = O_n(\varepsilon)$ (see (1.4) and (2.3)). In an asymptotically long time interval $t \sim \varepsilon^{-1}$ this state of motion breaks down and, in the general case, a considerable change occurs in all the osculating variables a_n, b_n, $n \geq 1$. The oscillations become multifrequency oscillations, which again leads to the above-mentioned basic difficulties in their asymptotic analysis and approximate calculation.

2. We will narrow the class of perturbing functions of system (1.1) and consider the case where the function f can be approximated by a polynomial of finite degree M of the variables u, u', $\dot{u}$ in a certain domain D

$$f(x,u,u',\dot{u}) = \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K} f_{ijk}(x)u^i(u')^j\dot{u}^k, \quad I+J+K = M \tag{3.2}$$

It follows from (3.2) that after substituting the function $u_{(1)}(x, a_1, b_1, \varphi_1)$ and its derivatives into f, the function $f_{(1)}$ obtained will be a trigonometric polynomial containing the frequencies $m\nu_1$ ($m = 0, 1, \ldots, M$)

$$f_{(1)}(x,a_1,b_1,\varphi_1) \equiv f(x,u_{(1)},u'_{(1)},\dot{u}_{(1)}), \quad \varphi_1 = \nu_1 t \tag{3.3}$$

$$u_{(1)} = X_1(x)(a_1\sin\varphi_1 + b_1\cos\varphi_1), \quad u'_{(1)} = \partial u_{(1)}/\partial x, \quad \dot{u}_{(1)} = \partial u_{(1)}/\partial t$$

The analytical properties of the function $f_{(1)}$ (3.3) and its mean can be established by elementary methods.

3. We will now introduce a "frequency" condition, imposed on the quantity ν_n, i.e. on the coefficients $\alpha_{0,1}$, $\beta_{0,1}$, which define the eigenfrequencies of the unperturbed system from (1.4). We will assume that

$$|m\nu_1-\nu_n|\geqslant\gamma>0,\quad m=0,1,\dots,M,\quad n=2,3,\dots \tag{3.4}$$

Then, assuming $\theta_n=\dot\theta_n\equiv 0$ for $n\geqslant 2$ in $f_1, f_2, \dots, f_n, \dots$, (2.2), we obtain expressions for F_n, which depend on a_1 and b_1 and are periodic in t with a frequency basis $\{m\nu_1\}$, $m=0,1,\dots,M$. As a result, the right-hand sides of the system allow of uniform averaging over t when $a_n=b_n=0$, $n\geqslant 2$, and by (3.4) we have

$$a_1^{\bullet}=\nu_1^{-1}F_1^c(a_1,b_1),\quad b_1^{\bullet}=-\nu_1^{-1}F_1^s(a_1,b_1),\quad a_n^{\bullet}=b_n^{\bullet}=0,\quad n\geqslant 2$$

$$a_1(0)=a_1^0,\quad b_1(0)=b_1^0,\quad a_n(0)=b_n(0)=0 \tag{3.5}$$

$$F_1^{c,s}(a_1,b_1)=\frac{1}{2\pi}\int_0^{2\pi}F_{(1)}(a_1,b_1,\varphi_1)\left\|\begin{matrix}\cos\varphi_1\\ \sin\varphi_1\end{matrix}\right\|d\varphi_1,\quad F_n^{c,s}\equiv 0$$

In (3.5) and below the dot over a quantity denotes the derivative with respect to the slow time $\tau=\varepsilon t$; the expressions for $F_{(n)}$ denote that we have substituted the quantity $a_n=b_n=0, n\geqslant 2$ into the functions F_n (2.3). We will assume that the solution $a^*{}_1(\tau)$, $b^*{}_1(\tau)$ of the averaged system of the first approximation (3.5) and the corresponding single-frequency approximation of the required solution $u_{(1)}$ are known by (2.1), (2.3) and (3.1)

$$a_1=a_1^*(\tau,a_1^0,b_1^0),\quad b_1=b_1^*(\tau,a_1^0,b_1^0),\quad a_n=b_n\equiv 0,\quad n\geqslant 2 \tag{3.6}$$

$$u_{(1)}^*(x,\tau,\varphi_1)=X_1(x)(a_1^*(\tau)\sin\varphi_1+b_1^*(\tau)\cos\varphi_1),\quad 0\leqslant t\leqslant L\varepsilon^{-1},\quad L=\text{const}$$

4. We will assume that the values of the function $u_{(1)}^*$ (3.6) and its derivatives $u_{(1)}^{*\prime}$, $\dot u_{(1)}^*$, see (3.3), belong to the domain $D_{(1)}\subset D$ together with a small neighbourhood; in addition, a Lipschitz constant $\lambda(D)$ exists such that

$$|f(x,u_{(1)},u'_{(1)},\dot u_{(1)})-f(x,u,u',\dot u)|\leqslant\lambda(|u_{(1)}-u|+|u'_{(1)}-u'|+|\dot u_{(1)}-\dot u|), \tag{3.7}$$

$$(u,u',\dot u)\in D,\quad (u_{(1)},u'_{(1)},\dot u_{(1)})\in D$$

5. We will assume that the function $f_{(1)}(x, a_1, b_1, \varphi_1)$ (3.3) with respect to the variable x belongs to a fairly high Steklov class of smoothness k [3]. This condition corresponds to the absence of perturbations of boundary conditions (1.2) and to a rapid decrease in $f_n\to 0$, $n\to\infty$.

In fact, suppose that, when $x=0,1$, the relations $f_{(1)}/\partial f_{(1)}/\partial x=\dots=\partial^{k-1}f/\partial x^{k-1}\equiv 0$ hold identically with respect to φ_1, and the derivative $\partial^k f_{(1)}/\partial x^k$ is piecewise-continuous. Then for $f_n(\varphi_1)$ we obtain, by integration by parts, the estimate $f_n=O(\nu_n^{-k})$, i.e. $f_n\sim n^{-k}$ uniformly with respect to φ_1.

4. AN ESTIMATE OF THE ACCURACY OF THE SINGLE-FREQUENCY APPROXIMATION

To validate the single-frequency approximation method, as it applies to problem (1.1)–(1.3), which satisfies the conditions listed in Section 3, we will consider the differences

$$\delta u=u-u_{(1)}^*,\quad \delta u'=u'-u_{(1)}^{*\prime},\quad \delta\dot u=\dot u-\dot u_{(1)}^* \tag{4.1}$$

Here $u_{(1)}^*$ is the solution of the first approximation, known from (3.6). The function $u=u(x,t,\varepsilon)$ is the unknown solution of the original initial-boundary-value problem, which can be represented in the form of an integral equation using Green's function G of the unperturbed problem

$$u=u^0(x,t)+\varepsilon\int_0^t\int_0^1 G(x,y,t-s)f(y,u,u',\dot u)dyds$$

$$u^0(x,t)=X_1(x)(a_1^0\sin\varphi_1+b_1^0\cos\varphi_1),\quad \varphi_1=\nu_1 t \tag{4.2}$$

$$G(x,y,t)=\sum_{n=1}^{\infty}X_n(x)X_n(y)\frac{\sin\nu_n t}{\nu_n}$$

By means of identity transformations, we can reduce the differences (4.1), using (4.2), to a form which is convenient for applying Gronwall's lemma

$$\begin{aligned}
&\delta u = \Delta u_{(1)}(x,t,\tau,\varepsilon)+\varepsilon\int_0^t\int_0^1 G(x,y,t-s)\Delta f_{(1)}dyds \\
&\Delta u_{(1)} \equiv u^0(x,t)+\varepsilon\int_0^t\int_0^1 G(x,y,t-s)f(y,\overset{*}{u}_{(1)},\overset{*}{u}{}'_{(1)},\dot{\overset{*}{u}}_{(1)})dyds-\overset{*}{u}_{(1)}(x,\tau,\varphi_1) \\
&\Delta f_{(1)} = f(y,u,u',\dot{u})-f(y,\overset{*}{u}_{(1)}(y,\sigma,\psi_1),\overset{*}{u}{}'_{(1)}(y,\sigma,\psi_1),\dot{\overset{*}{u}}_{(1)}(y,\sigma,\psi_1)) \\
&\delta u' = \partial\delta u/\partial x,\quad \delta\dot{u} = \partial\delta u/\partial t,\quad \sigma=\varepsilon s,\quad \psi_1 = \nu_1 s
\end{aligned} \tag{4.3}$$

For the known function $\Delta u_{(1)}$ and its derivatives with respect to x and t, by virtue of condition 5 in Section 3, we have the following limit with respect to the root mean-square norm $L_2(x)$

$$\|\Delta u_{(1)}\|+\|\Delta u'_{(1)}\|+\|\Delta \dot{u}_{(1)}\| \leqslant \varepsilon U,\quad 0\leqslant t\leqslant L\varepsilon^{-1} \tag{4.4}$$

Suppose $w = w(x, t)$ is the derivative of the function of $L_2(x)$ ($0 \leqslant x \leqslant 1$ and t is a parameter); then we have the following limit for it implied by the form of Green's function G (4.2)

$$\begin{aligned}
&\left\|\int_0^1 G(x,y,t-s)w(y,s)dy\right\|_{L_2} \leqslant A\|w\|_{L_2} \\
&\left\|\frac{\partial}{\partial x}\int_0^1 G(x,y,t-s)w(y,s)dy\right\|_{L_2} \leqslant B\|w\|_{L_2} \\
&\left\|\frac{\partial}{\partial t}\int_0^1 G(x,y,t-s)w(y,s)dy\right\|_{L_2} \leqslant C\|w\|_{L_2}
\end{aligned} \tag{4.5}$$

We will estimate the errors δu, $\delta u'$, $\delta\dot{u}$ with respect to the norm in $L_2(x)$ using Gronwall's lemma. Taking property (3.7) and the limits (4.4) and (4.5) into account we obtain

$$\begin{aligned}
&\|\delta u\|_{L_2}+\|\delta u'\|_{L_2}+\|\delta\dot{u}\|_{L_2} \leqslant \varepsilon U e^{\varepsilon Nt},\quad U=\text{const} \\
&N=\lambda(A+B+C),\quad 0\leqslant t\leqslant L\varepsilon^{-1},\quad L=\text{const}
\end{aligned} \tag{4.6}$$

The following limits of order ε for δu and its derivatives follow from (4.6) over an asymptotically long time interval

$$\begin{aligned}
&\|\delta u\|_{L_2} \leqslant \varepsilon U e^{\varepsilon NT},\quad \|\delta u\|_C = \max_{0\leqslant x\leqslant 1}|\delta u| \leqslant \varepsilon U e^{\varepsilon NT} \\
&\|\delta u'\|_{L_2} \leqslant \varepsilon U e^{\varepsilon NT},\quad \|\delta\dot{u}\|_{L_2} \leqslant \varepsilon U e^{\varepsilon NT},\quad 0\leqslant t\leqslant L\varepsilon^{-1}
\end{aligned} \tag{4.7}$$

Hence, by (4.7) for δu we also have a stronger uniform estimate with respect to x.

5. POSSIBLE EXTENSIONS OF THE SINGLE-FREQUENCY APPROACH

Results similar to the above can be obtained for perturbations of a more general form and with more general assumptions regarding the generating and required solutions.

1. The function f can be represented by a finite trigonometric polynomial in t

$$f=\sum_{p=1}^{P}[f_p^s(x,u,u',\dot{u})\sin\Omega_p t+f_p^c(x,u,u',\dot{u})\cos\Omega_p t],\quad \Omega_p\geqslant 0 \tag{5.1}$$

In (5.1) the functions f_p^s and f_p^c must have the structure of polynomials of type (3.2) in the arguments u, u' and $\dot{u}$. It is also assumed that the set of frequencies $\{\Omega_p\}$ of the external perturbations must satisfy the separability condition, similar to (3.4), namely

$$|m_p \nu_1 \pm \Omega_p - \nu_n| \geq \gamma > 0, \quad m_p = 0, 1, \ldots, M_p, \quad p = 1, \ldots, P \tag{5.2}$$

Here, as follows from (5.2), the conditions of resonance between the frequencies $m_p \nu_1$ and Ω_p can be satisfied (with an error $O(\varepsilon)$) or not satisfied, i.e. $|m_p \nu_1 - \Omega_p| = O(\varepsilon)$ or $|m_p \nu_1 - \Omega_p| \geq \eta > 0$, where $m_p = 1, \ldots, M_p + 1, p = 1, \ldots, P$.

2. When more restrictive requirements are imposed on the smoothness property of the exciting function of the form (5.1) the structure of its coefficients can be generalized as follows [4]:

$$f_p^{s,c} = f_p^{s,c}(x, u, u', u'', \dot{u}, \dot{u}', \dot{u}'') \tag{5.3}$$

The coefficients $f_p^{s,c}$ are functions of the polynomial type (3.2) in the argument $u, u', u'', \dot{u}, \dot{u}', \dot{u}''$ of degree M_p. It is assumed that the "single-frequency" condition (5.2) is satisfied, and also the assumption about the presence or absence of the external resonance. Note that terms containing u' and u'' (for example, of the form $u'^2 u''$ [10]), correspond to the geometrical non-linearity, $\dot{u}''$ corresponds to the internal dissipation, $\dot{u}$ corresponds to the external dissipation, while terms containing u may be due to the external elastic medium. A generalization of the structure of the function f (5.3) is of considerable interest from the applied point of view (see Section 7).

3. The single-frequency condition can be extended to the case where several lower modes of oscillations are excited, whereas subsequent modes satisfy the frequency separability condition of the form (5.2). The limiting conditions on the initial distributions $h(x)$ and $g(x)$ (1.3) have the form of relations which generalize (3.1)

$$h_q^2 + g_q^2 > 0, \quad q = 1, \ldots, Q, \quad h_n^2 + g_n^2 = O_n(\varepsilon^2), \quad n \geq Q + 1 \tag{5.4}$$

The system of the first approximation is of the order of $2Q$ and extends the equations of the single-frequency approximation (3.5); taking (5.4) into account we will represent the Cauchy problem in the form

$$\dot{a}_q = \nu_q^{-1} F_q^c(a_1, \ldots, a_Q, b_1, \ldots, b_Q), \quad \dot{b}_q = -\nu_q^{-1} F_q^s(a_1, \ldots, a_Q, b_1, \ldots, b_Q) \tag{5.5}$$

$$a_q(0) = a_q^0 = g_q \nu_q^{-1}, \quad b_g(0) = b_q^0 = h_q, \quad q = 1, \ldots, Q, \quad a_n = b_n \equiv 0, \quad n \geq Q + 1$$

The averaged functions $F_n^{c,s}$ (5.5) are obtained by substituting the expressions for θ_q, $\dot{\theta}_q$ (2.3) into $f_n(t, \theta, \dot{\theta})$ with $a_n = b_n = 0$ $(n \geq Q + 1)$ and averaging over the explicitly occurring argument t. It is assumed that the functions $F_n \cos \varphi_n$, $F_n \sin \varphi_n$ $(n \geq Q + 1)$ have a zero mean uniformly with respect to a_q and b_q: $F_n^{c,s} \equiv 0$; hence, in the first approximation it follows that $a_n = b_n \equiv 0$ (see (5.5)). The corresponding frequency condition, which generalizes (5.2), can be represented in the form

$$\left|\sum_{q=1}^{Q} m_{pq} \nu_q \pm \Omega_p - \nu_n\right| \geq \gamma > 0, \quad q = 1, \ldots, Q, \quad n \geq Q + 1$$
$$m_{pq} = 0, \pm 1, \ldots, \pm M_p, \quad \sum_{q=1}^{Q} |m_{pq}| \leq M_p, \quad p = 1, \ldots, P \tag{5.6}$$

As in Section 1, the resonance relation between the frequencies $m_{pq} \nu_q$ and can be satisfied or not. The solution of the Q-frequency approximation with error $O(\varepsilon)$ for $t \sim \varepsilon^{-1}$, according to representation (2.1) and on the basis of the constructions in Section 4, will be the function

$$u_{(Q)}(x, t, \tau) = \sum_{q=1}^{Q} X_q(x)(a_q^*(\tau) \sin \varphi_q + b_q^*(\tau) \cos \varphi_q) \tag{5.7}$$

where $a_q^*(\tau)$, $b_q^*(\tau)$ $(q = 1, \ldots, Q)$ is the solution of Cauchy problem (5.5).

4. The approach described above to the construction of an approximate solution can also be extended to the case of a system with slowly varying parameters [4]: $f = f(x, t, \tau, u, u', \dot{u}, \ldots)$. The use of the algorithm for constructing an approximate solution and the check of the conditions are considerably simplified when f is linear in $u, u', \dot{u}, \ldots$ (see Section 6.1).

5. As in the case of an equation of the form (1.1), we can similarly consider the problem of the Q-frequency approximation for the more general equation

$$\ddot{u} = u'' - \varkappa^2 u + \varepsilon f(x, t, \tau, u, u', \dot{u}, \ldots) \tag{5.8}$$

in which the term $\varkappa^2 u$ takes into account the effect of the external elastic medium. The presence of this term in problem (5.8), (1.2) somewhat changes the frequency ν_n of the natural oscillations of the unperturbed system, but their asymptotic form as $n \to \infty$ remains as before (see Section 1). This confirms the fact that the problem of internal resonance and of "small denominators" remains, and additional serious limitations on the structure of the function f and the frequency ν_n, similar to conditions (3.2), (5.1) and (5.6), are required.

6. With the corresponding requirements on the behaviour of the spectrum $\{\nu_n\}$, the method described in Section 2 can be used for an approximate finite-frequency analysis of the considerably inhomogeneous weakly non-linear system described by the initial-boundary-value problem of the form

$$\begin{gathered}
\rho(x)\ddot{u} = (p(x)u')' - r(x)u + \varepsilon f(x, t, \tau, u, u', \dot{u}, \ldots) \\
\alpha_0 p(0)u'(0,t) - \beta_0 u(0,t) = -\varepsilon \Phi_0(t, \tau, u(0,t), u'(0,t), \ldots) \\
\alpha_1 p(1)u'(1,t) + \beta_1 u(1,t) = \varepsilon \Phi_1(t, \tau, u(1,t), u'(1,t), \ldots) \\
u(x,0) = h(x), \quad \dot{u}(x,0) = g(x), \quad 0 < x < 1 \\
p(x) \geqslant p_0 > 0, \quad \rho(x) \geqslant \rho_0 > 0, \quad r(x) \geqslant 0
\end{gathered} \tag{5.9}$$

Here $\rho(x)$ is the linear density, $p(x)$ is the stiffness per unit length, $r(x)$ is the coefficient of elasticity of the external medium and $\Phi_{0,1}$ are certain functions of polynomial structure, similar to f. For highly accurate calculations of the natural frequencies and forms of the oscillations of the unperturbed system (5.9) one can use well-developed effective numerical-analytic methods of accelerated convergence [11, 12].

It should be noted that the system of equations of the type (2.2) for generalized coordinates $\theta_n = \theta_n(t, \varepsilon)$ in this case takes into account small non-linear perturbations of the boundary conditions

$$\begin{gathered}
\ddot{\theta}_n + \nu_n^2 \theta_n = \varepsilon f_n(t, \tau, \theta, \dot{\theta}) + \varepsilon \Psi_n(t, \tau, \theta, \dot{\theta}) \\
\Psi_n = \frac{p(1)}{\beta_1} X_n'(1)\Phi_1^*(t, \tau, \theta, \dot{\theta}) - \frac{p(0)}{\beta_0} X_n'(0)\Phi_0^*(t, \tau, \theta, \dot{\theta})
\end{gathered} \tag{5.10}$$

The functions $\Phi_{0,1}^*$ in (5.10) denote that the expressions for $u(x, t)$ $\dot{u}(x, t)$ (and their derivatives) when $x = 0, 1$ can be represented in the form (2.1): $(X(\theta), \theta)$, $(X(1), \theta)$, $(X(0), \dot{\theta})$, $(X1), \dot{\theta})$ (and similarly for the derivatives u', u'', $\dot{u}'$, $\dot{u}''$,). Note also that when $\beta_0 = 0$ ($\alpha_0 = 1$) or (and) $\beta_1 = 0$ ($\alpha_1 = 1$) in the expressions for Ψ_n (5.10) the transformations $(p(0)/\beta_0) X'_n(0) = X_n(0)/\alpha_0$ or (and) $(p(1)/\beta_1) X'_n(1) = -X_n(1)/\alpha_1$ are carried out, which enable the singularities to be eliminated.

It follows from the expression for Ψ_n (5.10) that in the general case, condition 5* in Section 3 breaks down, since $\Psi_n \sim \nu_n \sim n$. Hence, for rapid convergence of the Fourier series it is necessary to satisfy the smoothness condition $\Phi_{0,1} \equiv 0$. Hence, the requirement for the boundary conditions to be linear and homogeneous is essential in order to validate the convergence of the series and the smallness of the error of the solution and of its derivatives

7. In addition to the case of a scalar variable u we can consider the more general situation of a vector function $u = (u_1, u_2, \ldots, u_S)$, each of the components of which is described by an initial-boundary-value problem of the form (5.9), (1.2) and (5.5), and the relation between the components is provided by the perturbations of f and $\Phi_{0,1}$ (for example, when investigating three-dimensional non-linear vibration of a string [10], see Section 7).

6. PERTURBATIONS OF SPECIAL FORM

We will describe the finite-mode approximation method in some special cases, which enable an approximate solution to be constructed completely.

6.1. *A linear perturbation*. Consider problem (5.9), assuming the function f to be linear with respect to the unknown u and its derivatives $\Phi_{0,1} \equiv 0$. We have the following expression for f

$$f = F(x,t,\tau) + A(x,\tau)u + B(x,\tau)u' + C(x,\tau)u'' + \\ +E(x,\tau)\dot{u} + R(x,\tau)\dot{u}' + H(x,\tau)\dot{u}'' \tag{6.1}$$

We will construct the solution $u(x, t, \varepsilon)$ in the form of series (2.1) using the method described in Section 2. Suppose $\nu_n, X_n(x)$ is the known solution of the unperturbed boundary-value problem for eigenvalues and functions [11, 12]

$$(p(x)X')' + (\lambda\rho(x) - r(x))X = 0, \quad 0 < x < 1 \\ \alpha_0 p(0)X'(0) - \beta_0 X(0) = 0, \quad \alpha_1 p(1)X'(1) + \beta_1 X(1) = 0 \tag{6.2}$$

where $\{X_n(x)\}$ is a system of eigenfunctions, orthonormalized with weight $\rho(x)$. Then, using (2.2) we obtain the expressions

$$f_n(t,\tau,\theta,\dot{\theta}) = F_n(t,\tau) + \sum_{m=1}^{\infty}(U_{nm}(\tau)\theta_m + V_{nm}(\tau)\dot{\theta}_m)$$

$$U_{nm}(\tau) = \int_0^1 X_n(x)(A(x,\tau)X_m(x) + B(x,\tau)X'_m(x) + C(x,\tau)X''_m(x))dx \tag{6.3}$$

$$V_{nm}(\tau) = \int_0^1 X_n(x)(E(x,\tau)X_m(x) + R(x,\tau)X'_m(x) + H(x,\tau)X''_m(x))dx$$

We now substitute the expressions for θ_n, $\dot{\theta}_n$ into f_n (6.3) using (2.3); we obtain a denumerable system of equations in the osculating variables a_n and b_n. Averaging over the explicitly occurring fast time t, we obtain a formal denumerable system of the first approximation in the slow time

$$\dot{a}_n = \tfrac{1}{2}V_{nn}(\tau)a_n + \tfrac{1}{2}\nu_n^{-1}U_{nn}(\tau)b_n + \nu_n^{-1}F_n^c(\tau), \quad a_n(0) = a_n^0 \\ \dot{b}_n = -\tfrac{1}{2}\nu_n^{-1}U_{nn}(\tau)a_n + \tfrac{1}{2}V_{nn}(\tau)b_n - \nu_n^{-1}F_n^s(\tau), \quad b_n(0) = b_n^0 \\ F_n^{c,s}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T F_n(t,\tau)\begin{Vmatrix}\cos\nu_n t \\ \sin\nu_n t\end{Vmatrix}dt, \quad n = 1,2,\ldots \tag{6.4}$$

Note that Eqs (6.4) have a certain structure and are uncorrelated for different values of n. The coefficients $-1/2V_{nn}$ characterize the partial dissipation, while $1/2\nu_n^{-1}U_{nn}$ characterize the additional effect of elasticity. We can obtain a complete analytic solution of the Cauchy problem in the form of quadratures of the unknown functions

$$\begin{Vmatrix} a_n^*(\tau) \\ b_n^*(\tau)\end{Vmatrix} = \Pi(\chi_n(\tau))\begin{Vmatrix} a_n^0 \\ b_n^0\end{Vmatrix}\exp\gamma_n(\tau) + \frac{1}{\nu_n}\int_0^\tau \Pi(\Delta\chi_n)\begin{Vmatrix} F_n^c(\sigma) \\ -F_n^s(\sigma)\end{Vmatrix}\exp\Delta\gamma_n(\tau,\sigma)d\sigma \\ \chi_n(\tau) = \frac{1}{2\nu_n}\int_0^\tau U_{nn}(\sigma)d\sigma, \quad \gamma_n(\tau) = \frac{1}{2}\int_0^\tau V_{nn}(\sigma)d\sigma, \quad \Pi^{-1}(\chi_n) = \Pi^T(\chi_n) \\ \Delta\chi_n(\tau,\sigma) = \chi_n(\tau) - \chi_n(\sigma), \quad \Delta\gamma_n(\tau,\sigma) = \gamma_n(\tau) - \gamma_n(\sigma) \tag{6.5}$$

Here $\Pi(\chi_n)$ is the 2×2 matrix of rotation by an angle $\varkappa_n$. The formal solution of the first approximation $u_{(1)}(x, t, \tau)$ is obtained after substituting the slow functions $a_n^*(\tau)$, $b_n^*(\tau)$ into the expression for θ_n and then into (2.1).

We will require that the Fourier coefficients of the initial distributions for the displacements $h_n = \langle h, X_n\rangle_\rho = 0$ and the velocities $g_n = \langle g, X_n\rangle_\rho = 0$ and, in addition, $F_n^{c,s}(\tau) \equiv 0$ when $n \geqslant Q + 1$. Then, when there is a sufficiently rapid decrease in the coefficients $U_{nq}(\tau)$, $V_{nq}(\tau)$ when $n \to \infty$ for all $q = 1, 2, \ldots, Q$, $|\tau| \leqslant L$, we obtain the non-formal finite-mode solution

$$u_{(Q)}(x,t,\tau) = \sum_{q=1}^{Q} X_q(x)(a_q^*(\tau)\sin\nu_q t + b_q^*(\tau)\cos\nu_q t) \tag{6.6}$$

The coefficient $U_{nq}(\tau)$, $V_{nq}(\tau)$ will decrease at a rate $\nu_n^k \sim n^{-k}$ if the functions A, B, C, E, R and H possess smooth derivatives with respect to x up to the $(k-1)$th order inclusive, which vanish when $x = 0$ and $x = 1$, while the kth derivative is piecewise-continuous [3] (see condition 5 of Section 3).

6.2. *A quasi-conservative perturbation*. We will assume that the function f does not contain time and the derivative $\dot{u}, \dot{u}', \dot{u}''$ explicitly, i.e. it has the form of a polynomial of degree M in u, u', u''. Suppose also that the conditions of the single-frequency approximation (3.1) and (3.4) are satisfied. Then, the averaged equations for a_1 and b_1, by (3.5), have, in the slow time τ, the form of a time-varying system

$$\dot{a}_1 = \nu_1^{-1}F_1^c(\tau, a_1, b_1), \quad \dot{b}_1 = -\nu_1^{-1}F_1^s(\tau, a_1, b_1), \quad a_n, b_n \equiv 0, \quad n \geq 2$$
$$F_1^{c,s}(\tau, a_1, b_1) = \frac{1}{2\pi}\int_0^{2\pi} F_{(1)}(\tau, a_1 \sin\varphi_1 + b_1 \cos\varphi_1)\left\| \begin{matrix} \cos\varphi_1 \\ \sin\varphi_1 \end{matrix} \right\| d\varphi_1 \tag{6.7}$$
$$F_{(1)}(\tau, \theta_1) = \int_0^1 f(x, \tau, X_1(x)\theta_1, \ X_1'(x)\theta_1, \ X_1''(x)\theta_1)X_1(x)dx$$

Note the structural property of system (6.7), which enables the equations to be completely integrated. Multiplying the first equation by a_1 and the second by b_1 and adding them, we obtain $(\dot{a}_1^2 + \dot{b}_1^2) = 0$, i.e. the amplitude of the first mode $r_1 = (a_1^2 + b_1^2)^{1/2} = \text{const}$ and is determined by the initial values $a_1^0 + b_1^0$. We will represent the required functions a_1 and b_1 in the form $a_1 = r_1 \cos\psi_1$, $b_1 = r_1 \sin\psi_1$. Differentiating with respect to τ and using (6.7) we obtain the following expression for the unknown phase ψ_1

$$\psi_1(\tau) = \psi_1^0 - \frac{1}{r_1\nu_1}\int_0^\tau F_1^s(\sigma, r_1, 0)d\sigma, \quad \cos\psi_1^0 = \frac{a_1^0}{r_1}, \quad \sin\psi_1^0 = \frac{b_1^0}{r_1} \tag{6.8}$$

Hence, in the first single-mode approximation, the amplitude of the oscillations r_1 and the energy $1/2(\nu_1^2\theta_1^2 + \dot{\theta}_1^2) = 1/2\nu_1^2 r_1^2$ are conserved with an error $O(\varepsilon)$ in a time interval $t \sim \varepsilon^{-1}$. The phase of the oscillations is found from (6.8); we finally obtain $\theta_1 = r_1 \sin(\nu_1 t + \psi_1(\tau))$.

7. THREE-DIMENSIONAL VIBRATION OF A NON-UNIFORM STRING

We will consider the three-dimensional non-linear vibration of a nonuniform string with fixed ends. We will take into account the extensibility of the thread, and also the forces of linear dissipation and distributed forces applied from the external medium. We will derive the equations and analyse the forced steady vibration by analogy with the case of a uniform string [10, 13, 14]. Taking into account the last terms of the expansion in the expression for the potential energy of elastic deformation and the extensibility, we obtain an initial-boundary-value problem for the two-dimensional vector with boundary conditions of the first kind

$$\rho(x)\ddot{\mathbf{u}} = \mathrm{T}\mathbf{u}'' - \mathrm{K}(x)\dot{\mathbf{u}} + \tfrac{1}{8}(P(x)\partial\mathbf{u}'^4/\partial\mathbf{u}')' + \mathbf{\Phi}(x,t), \quad 0 < x < l, \quad t \geq 0$$
$$\mathbf{u}(0,t) = \mathbf{u}(l,t) = 0, \quad \mathbf{u}(x,0) = \mathbf{h}(x), \quad \dot{\mathbf{u}}(x,0) = \mathbf{g}(x) \tag{7.1}$$

Here l is the length, $\rho(x) = dS(x)$ is the linear density of the string, d is the volume density, $S(x)$ is the cross-section area, and T is the tensile force of the thread; these characteristics correspond to the undeformed state. Further, $P(x) = ES(x) - \mathrm{T}$, where E is Young's modulus, Φ is the distributed external force, periodic with respect to t, and $K(x)$ is the coefficient of external linear friction per unit length. The vector $\mathbf{u}$ describes the displacements of a point on the string in the yz plane, orthogonal to its undeformed state; the ends of the string are rigidly clamped. The quantity u'^4 in (7.1) is defined in the usual way: $\mathbf{u}'^4 = (y'^2 + z'^2)^2$; the derivative with respect to the vector $\mathbf{u}'$ is also understood in the usual sense. Note that the linear density depends on the value of the deformations $\mathbf{u}'$, due to the change in the length of the string. However, because of the strong inequality $ES \gg \mathrm{T}$, which holds for actual materials (usually $ES/\mathrm{T} \sim 10^2$–10^3), this change can be neglected and, moreover, the quantity T can be dropped in the expression for $P(x)$.

Suppose S_0 is the characteristic value of $S(x)$, where $S/S_0 \sim 1$ for all $0 < x < l$; u_0 is the maximum displacement $\mathbf{u}(x, t)$, i.e. $|\mathbf{u}| \leqslant u_0$. We will introduce dimensionless variables, taking the quantity l as the unit of length and the quantity ν_*^{-1} as the unit of time, where $\nu_*^2 = T(l^2 dS_0)^{-1}$. We will denote by K_0 the maximum value of $K(x)$, and by Φ_0 the maximum of $|\Phi|$ with respect to x, t and we will assume that the corresponding perturbing terms, namely, the geometrical non-linearity, the dissipation and the external force, are relatively small [10, 13]. Then, system (7.1) can be reduced to a form which enables us to use the asymptotic approach. For clarity we will represent the equations of the three-dimensional vibration in coordinate form, apart from terms $O(\varepsilon^2)$

$$\begin{gathered} r(x)\ddot{y} = y'' + (\varepsilon/2)[r(x)(y'^3 + y'z'^2)]' - \varepsilon\varkappa(x)\dot{y} + \varepsilon F_y(x,t) \\ r(x)\ddot{z} = z'' + (\varepsilon/2)[r(x)(y'^2 z' + z'^3)]' - \varepsilon\varkappa(x)\dot{z} + \varepsilon F_z(x,t) \\ 0 < x < 1, \quad 0 < \varepsilon \ll 1, \quad r(x) = S(x)/S_0, \quad (ES_0/T)(u_0/l)^2 = \varepsilon \\ \varepsilon\varkappa(x) \equiv K(x)(\nu_* dS_0)^{-1}, \quad \varepsilon\mathbf{F}(x,t) \equiv \mathbf{\Phi}(x,t)l^2(Tu_0)^{-1} \end{gathered} \tag{7.2}$$

The boundary conditions at $x = 0, 1$ have the form (7.1), while the initial distributions of h and g are reduced to dimensionless variables.

We will assume that the boundary-value problems for Eqs (7.2) when $\varepsilon = 0$ can be fairly completely investigated, i.e. the solution of the Sturm–Liouville problem of type (6.2) is known

$$X'' + \lambda r(x)X = 0, \quad X(0) = X(1) = 0, \quad \{\lambda_n\}, \quad \{X_n(x)\}$$

Here λ_n are the eigenvalues and $X_n(x)$ are eigenfunctions, orthonormalized with weight $r(x)$. This solution can be effectively constructed using the method of accelerated (quadratic) convergence [11, 12]. Using the procedure employed in Section 2 we obtain the equations of the single-mode approximation (a Duffing type vector equation)

$$\begin{gathered} \ddot{\mathbf{s}}_1 + \lambda_1 \mathbf{s}_1 = -\varepsilon\gamma_1 s_1^2 \mathbf{s}_1 - \varepsilon\sigma_1\dot{\mathbf{s}}_1 + \varepsilon\mathbf{f}_1(\Omega t) \quad (\mathbf{f}_1 = \mathbf{f}^* \cos\Omega t), \quad \mathbf{u}_{(1)} = X_1\mathbf{s}_1 \\ \gamma_1 = \frac{1}{2}\int_0^1 X_1'^4(x)r(x)dx, \quad \sigma_1 = \int_0^1 X_1(x)\varkappa(x)dx, \quad \mathbf{f}_1(\Omega t) \equiv \int_0^1 X_1(x)\mathbf{F}(x,t)dx \end{gathered} \tag{7.3}$$

Small-parameter methods [4–6, 8] are applicable to the quasi-linear oscillatory system (7.3). Free vibration ($\mathbf{f}_1 \equiv 0$) and forced vibration (in the ε-neighbourhood of the principal resonance $\Omega = \nu_1$),

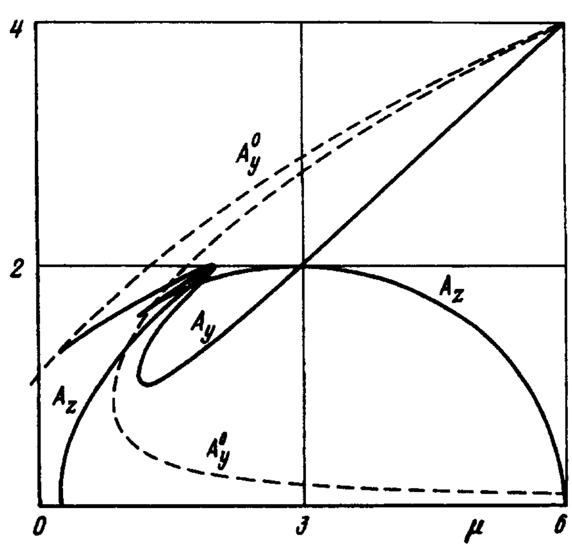

Fig. 1.

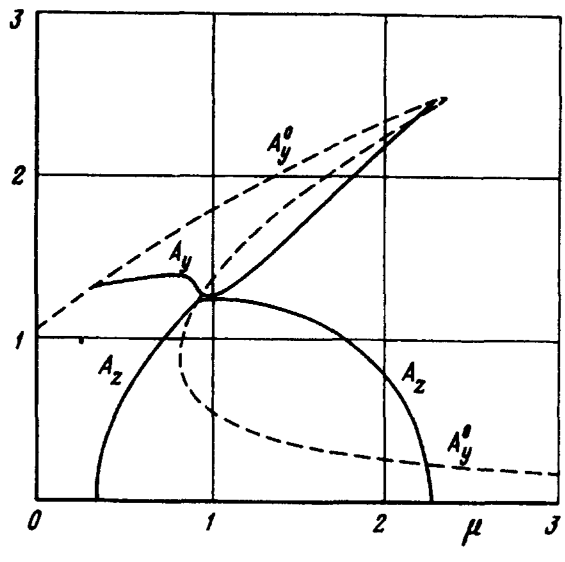

Fig. 2.

ignoring dissipation ($\sigma_1 = 0$), have been investigated in detail in [10, 13]. Resonance curves for the steady-state oscillations were constructed and analysed; the Lyapunov stability was investigated in the extremely interesting case where the external excitation only acts in one of the planes, while the vibration in the other plane has a parametric form.

For an adequate interpretation of the results of experiments [13] it is necessary to take dissipation into account, for example, using model (7.3) with $\sigma_1 > 0$. The corresponding resonance curves can be reduced, by scale transformations, to a single-parameter family [14]. It is convenient to take σ_1 as the parameter of the family, while the parameters $\lambda_1 = \gamma_1 = f_y^* = 1, f_z^* = 0$. Typical curves of the amplitudes of plane vibration $A_y^0(A_z^0 \equiv 0)$ and three-dimensional vibration A_y and A_z as a function of the frequency-detuning parameter $\mu = (\Omega - 1)\varepsilon^{-1} > 0$ are shown in Figs 1 and 2 for $\sigma_1 = 0.25$ and $\sigma_1 = 0.4$; they have an exotic form. Note that $A_z > 0$ over a certain range of variation of $\mu > 0$, which depends on σ_1. For fairly large values of $\sigma_1 > 3^{1/2}4^{-5/6} \simeq 0.546$, steady vibration in the xz plane is impossible [14]. Note that if internal dissipation of the type $H(x)\mathbf{u}''$ is taken into account the same results are obtained; then σ_1 is the total coefficient of linear dissipation with respect to the first mode.

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